

CONNES INTEGRATION FORMULA FOR THE NONCOMMUTATIVE PLANE

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ABSTRACT. Our aim is to prove the integration formula on the noncommutative (Moyal) plane in terms of singular traces *a la* Connes.

1. INTRODUCTION

Let M be a compact Riemannian manifold. The following formula can be found in p. 34 in [1] and in Corollary 7.21 in [9].

$$(1) \quad \mathrm{Tr}_\omega(M_f(1 - \Delta)^{-\frac{d}{2}}) = \int_M f d\mathrm{vol}, \quad f \in C^\infty(M).$$

Here, M_f is the multiplication operator, Δ is the Hodge-Laplacian operator on $L_2(M, \mathrm{vol})$ and Tr_ω is the Dixmier trace on the ideal $\mathcal{L}_{1,\infty}$ (see Section 2). Also, Corollary 7.22 in [9] wrongly extends this result to $f \in L_1(M, \mathrm{vol})$ (in fact, $f \in L_2(M, \mathrm{vol})$ is the necessary and sufficient condition for this formula to hold; see [14] or the book [15] for detailed proofs).

According to [1], formula (1) “led Connes to introduce the Dixmier trace as the correct operator theoretical substitute for integration of infinitesimals of order one in non-commutative geometry.” It appears suitable to refer to (1) and similar results as the “Connes Integration Formula”.

Compactness of the (resolvent of the) Hodge-Dirac operator plays a crucial role in the proofs of Connes Integration Formula for unital spectral triples (see [1] and [9]). For non-unital spectral triples (including non-compact manifolds), the proofs become radically harder. Even the case of the simplest non-compact manifold \mathbb{R}^d required a substantial effort and the first reasonable answer was very recently given in [11] (see the book [15] for detailed proofs).

In this paper, we investigate the validity of Connes Integration Formula for the noncommutative (Moyal) plane \mathbb{R}_θ^d (here, θ is a non-degenerate antisymmetric matrix). Earlier attempts in this direction can be found in [8] (see Proposition 4.17 there), [2] and [3]. We substantially strengthen corresponding results from these papers and present a completely different approach to Connes Integration Formula. The novelty of our approach is in the consistent use of Cwikel estimates for the noncommutative plane (obtained in a recent paper [12]) — see Section 2.

Our main result is the following theorem.

Theorem 1.1. *If $x \in W^{d,1}(\mathbb{R}_\theta^d)$, then $x(1 - \Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$ and*

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \tau_\theta(x)$$

for every normalised continuous trace φ on $\mathcal{L}_{1,\infty}$.

Here, $W^{d,1}(\mathbb{R}_\theta^d)$ is a Sobolev space on \mathbb{R}_θ^d and τ_θ is the faithful normal semifinite trace on $L_\infty(\mathbb{R}_\theta^d)$.

Section 2 involves the preliminaries necessary to prove Theorem 1.1. In Section 3, we prove that

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = c_\varphi \tau_\theta(x), \quad x \in W^{d,1}(\mathbb{R}_\theta^d),$$

for every normalised trace on $\mathcal{L}_{1,\infty}$. In Section 4, we construct *one particular* $x \in W^{d,1}(\mathbb{R}_\theta^d)$ such that $\varphi(x(1 - \Delta)^{-\frac{d}{2}})$ does not depend on the choice of a normalised continuous trace φ . The combination of these results yield Theorem 1.1.

2. PRELIMINARIES

2.1. General notation. Fix throughout a separable infinite dimensional Hilbert space H . We let $\mathcal{L}(H)$ denote the algebra of all bounded operators on H . For a compact operator T on H , let $\mu(k, T)$ denote k -th largest singular value (these are the eigenvalues of $|T|$). The sequence $\mu(T) = \{\mu(k, T)\}_{k \geq 0}$ is referred to as the singular value sequence of the operator T . The standard trace on $\mathcal{L}(H)$ is denoted by Tr .

Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in l_\infty$, we denote the corresponding diagonal operator by $\text{diag}(\alpha)$.

2.2. Schatten ideals \mathcal{L}_p and $\mathcal{L}_{p,\infty}$, $p > 0$. For every $p > 0$, we set

$$\mathcal{L}_p = \{T \in \mathcal{L}(H) : \text{Tr}(|T|^p) < \infty\}.$$

We set

$$\|T\|_p = (\text{Tr}(|T|^p))^{\frac{1}{p}}, \quad T \in \mathcal{L}_p.$$

For every $p > 0$, $\|\cdot\|_p$ is a quasi-norm¹ and $(\mathcal{L}_p, \|\cdot\|_p)$ is a quasi-Banach space. For $p \geq 1$, $\|\cdot\|_p$ is a norm. For $p < 1$, the space $(\mathcal{L}_p, \|\cdot\|_p)$ is not Banach — that is, its quasi-norm is not equivalent to any norm.

For a given $0 < p \leq \infty$, we let $\mathcal{L}_{p,\infty}$ denote the principal ideal in $\mathcal{L}(H)$ generated by the operator $\text{diag}(\{(k+1)^{-\frac{1}{p}}\}_{k \geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{T \in \mathcal{L}(H) : \mu(k, T) = O((k+1)^{-1/p})\}.$$

We set

$$\|T\|_{p,\infty} = \sup_{k \geq 0} (k+1)^{1/p} \mu(k, T), \quad T \in \mathcal{L}_{p,\infty}.$$

For every $p > 0$, $\|\cdot\|_{p,\infty}$ is a quasi-norm and $(\mathcal{L}_{p,\infty}, \|\cdot\|_{p,\infty})$ is a quasi-Banach space. For $p > 1$, $\|\cdot\|_{p,\infty}$ is equivalent to a (unitarily invariant Banach) norm. For $p \leq 1$, the space $(\mathcal{L}_{p,\infty}, \|\cdot\|_{p,\infty})$ is not Banach — that is, its quasi-norm is not equivalent to any norm. In [17], the Banach envelope of $\mathcal{L}_{1,\infty}$ was thoroughly investigated.

¹A quasinorm satisfies the norm axioms, except that the triangle inequality is replaced by $\|x+y\| \leq K(\|x\| + \|y\|)$ for some uniform constant $K > 1$.

2.3. Traces on $\mathcal{L}_{1,\infty}$.

Definition 2.1. *If \mathcal{I} is an ideal in $\mathcal{L}(H)$, then a unitarily invariant linear functional $\varphi : \mathcal{I} \rightarrow \mathbb{C}$ is said to be a trace.*

Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in \mathcal{L}(H)$, and since the unitaries span $\mathcal{L}(H)$, it follows that traces are precisely the linear functionals on \mathcal{I} satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in \mathcal{L}(H).$$

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace which is denoted $[\mathcal{I}, \mathcal{L}(H)]$ and defined to be the linear span of all commutators $[T, S] : T \in \mathcal{I}, S \in \mathcal{L}(H)$. It is shown in Lemma 5.2.2 in [15] that $\varphi(T_1) = \varphi(T_2)$ whenever $0 \leq T_1, T_2 \in \mathcal{I}$ are such that the singular value sequences $\mu(T_1)$ and $\mu(T_2)$ coincide.

For $p > 1$, the ideal $\mathcal{L}_{p,\infty}$ does not admit a non-zero trace [7], while for $p = 1$, there exists a plethora of traces on $\mathcal{L}_{1,\infty}$ (see e.g. [18] or [15]). A standard example of a trace on $\mathcal{L}_{1,\infty}$ is a Dixmier trace introduced in [6] that we now explain.

Definition 2.2. *Let ω be a free ultrafilter on \mathbb{Z}_+ . The functional*

$$\mathrm{Tr}_\omega : A \rightarrow \lim_{n \rightarrow \omega} \frac{1}{\log(2+n)} \sum_{k=0}^n \mu(k, A), \quad 0 \leq A,$$

is finite and additive on the positive cone of $\mathcal{L}_{1,\infty}$. Therefore, it extends to a trace on $\mathcal{L}_{1,\infty}$. We call such traces Dixmier traces.

These traces clearly depend on the choice of the ultrafilter ω on \mathbb{Z}_+ . Using a slightly different definition, this notion of trace was applied by Connes [4] in noncommutative geometry.

An extensive discussion of traces, and more recent developments in the theory, may be found in [15] including a discussion of the following facts. We refer the reader to an alternative approach to the theory of traces on $\mathcal{L}_{1,\infty}$ suggested in [18] (based on the fundamental paper [16] by Pietsch).

- (1) All Dixmier traces on $\mathcal{L}_{1,\infty}$ are positive.
- (2) All positive traces on $\mathcal{L}_{1,\infty}$ are continuous in the quasi-norm topology.
- (3) There exist positive traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces (see [18]).
- (4) There exist traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous (see [15]).

Definition 2.3. *We say that an operator $A \in \mathcal{L}_{1,\infty}$ is measurable if $\varphi(A)$ does not depend on the choice of the continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.*

2.4. Noncommutative plane: algebra. Each assertion in this subsection is rigorously established in Section 6 in [12].

Our approach to the noncommutative plane is to introduce the von Neumann algebra generated by a strongly continuous family of unitary operators $\{U(t)\}_{t \in \mathbb{R}^d}$, $d \in \mathbb{N}$, satisfying the commutation relation

$$(2) \quad U(t+s) = \exp\left(-\frac{i}{2}\langle t, \theta s \rangle\right) U(t)U(s), \quad t, s \in \mathbb{R}^d,$$

where θ is a fixed antisymmetric real $d \times d$ matrix. Namely, we set

$$(3) \quad (U(t)\xi)(u) = e^{-\frac{i}{2}\langle t, \theta u \rangle} \xi(u-t), \quad \xi \in L_2(\mathbb{R}^d), \quad u, t \in \mathbb{R}^d.$$

Definition 2.4. Let $d \in \mathbb{N}$ and let θ be a fixed non-degenerate² antisymmetric real $d \times d$ matrix. The von Neumann subalgebra in $\mathcal{L}(L_2(\mathbb{R}^d))$ generated by $\{U(t)\}_{t \in \mathbb{R}^d}$, introduced in (3), is called the noncommutative plane and denoted by $L_\infty(\mathbb{R}_\theta^d)$.

Example 2.5. If $d = 2$, then $L_\infty(\mathbb{R}_\theta^d)$ is generated by 2 unitary groups $t \rightarrow U_1(t)$, $t \rightarrow U_2(t)$, $t \in \mathbb{R}$ satisfying the condition

$$U_1(t_1)U_2(t_2) = e^{i\alpha t_1 t_2} U_2(t_2)U_1(t_1), \quad t_1, t_2 \in \mathbb{R}.$$

Here, $U_1(t_1) = U((t_1, 0))$ and $U_2(t_2) = U((0, t_2))$.

The following assertion is well-known. In [12], a *spatial* isomorphism is constructed.

Theorem 2.6. For every non-degenerate antisymmetric real matrix θ , the algebra $L_\infty(\mathbb{R}_\theta^d)$ is isomorphic to $\mathcal{L}(L_2(\mathbb{R}^{\frac{d}{2}}))$.

Having established the isomorphism between $r : L_\infty(\mathbb{R}_\theta^d) \rightarrow \mathcal{L}(L_2(\mathbb{R}^{\frac{d}{2}}))$ we now equip $L_\infty(\mathbb{R}_\theta^d)$ with a faithful normal semifinite trace $\tau_\theta = \text{Tr} \circ r$.

We can now define L_p -spaces on $L_\infty(\mathbb{R}_\theta^d)$.

$$L_p(\mathbb{R}_\theta^d) = \left\{ x \in L_\infty(\mathbb{R}_\theta^d) : \tau_\theta(|x|^p) < \infty \right\}.$$

Lemma 2.7. An operator $x \in L_\infty(\mathbb{R}_\theta^d)$ is in $L_2(\mathbb{R}_\theta^d)$ if and only if³

$$x = \text{Op}(f) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{d/4}} \int_{\mathbb{R}^d} f(s) U(s) ds$$

for some unique $f \in L_2(\mathbb{R}^d)$ with $\|x\|_2 = \|f\|_2$.

Note that our picture is the Fourier dual of the one considered in [8]. More precisely, the paper [8] deals with operators of the form $\text{Op}(\mathcal{F}f)$, where f is Schwartz (in [8], these operators are written simply as f).

2.5. Noncommutative plane: calculus. Each assertion in this subsection is rigorously established in Section 6 in [12].

Let D_k , $1 \leq k \leq d$ be multiplication operators on $L_2(\mathbb{R}^d)$

$$(D_k \xi)(t) = t_k \xi(t), \quad \xi \in L_2(\mathbb{R}^d).$$

For brevity, we denote $\nabla = (D_1, \dots, D_d)$. For every $1 \leq k \leq d$, we have

$$(4) \quad [D_k, U(s)] = s_k U(s), \quad s \in \mathbb{R}^d.$$

Moreover, we have

$$(5) \quad e^{i\langle t, \nabla \rangle} U(s) e^{-i\langle t, \nabla \rangle} = e^{i\langle t, s \rangle} U(s), \quad s, t \in \mathbb{R}^d.$$

If $[D_k, x] \in \mathcal{L}(L_2(\mathbb{R}^d))$ for some $x \in L_\infty(\mathbb{R}_\theta^d)$, then $[D_k, x] \in L_\infty(\mathbb{R}_\theta^d)$. This crucial fact allows us to introduce mixed partial derivative $\partial^\alpha x$ of $x \in L_\infty(\mathbb{R}_\theta^d)$.

²A non-degenerate antisymmetric matrix is automatically of even order.

³To be precise,

$$x = \lim_{N \rightarrow \infty} \frac{1}{(2\pi)^{d/4}} \int_{[-N, N]^d} f(s) U(s) ds,$$

where the limit is taken in $L_2(\mathbb{R}_\theta^d)$. In what follows, we write the integral over \mathbb{R}^d instead of the limit in order to lighten the notations.

Definition 2.8. Let α be a multiindex and let $x \in L_\infty(\mathbb{R}_\theta^d)$. If every repeated commutator $[D_{\alpha_j}, [D_{\alpha_{j+1}}, \dots, [D_{\alpha_n}, x]]]$, $1 \leq j \leq n$, is a bounded operator on $L_2(\mathbb{R}^d)$, then the mixed partial derivative $\partial^\alpha x$ of x is defined as

$$\partial^\alpha x = [D_{\alpha_1}, [D_{\alpha_2}, \dots, [D_{\alpha_n}, x]]].$$

In this case, we have that $\partial^\alpha x \in L_\infty(\mathbb{R}_\theta^d)$. As usual, $\partial^0 x = x$.

Therefore, we can introduce the Sobolev space $W^{m,p}(\mathbb{R}_\theta^d)$ associated with the noncommutative plane in the following way.

Definition 2.9. For $m \in \mathbb{Z}_+$ and $p \geq 1$, the space $W^{m,p}(\mathbb{R}_\theta^d)$ is the space of $x \in L_p(\mathbb{R}_\theta^d)$ such that every partial derivative of x up to order m is also in $L_p(\mathbb{R}_\theta^d)$. This space is equipped with the norm,

$$\|x\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\partial^\alpha x\|_p, \quad x \in W^{m,p}(\mathbb{R}_\theta^d).$$

The following assertion is one of the main results in [12].

Theorem 2.10. If $x \in W^{d,1}(\mathbb{R}_\theta^d)$, then

(a) $x(1 - \Delta)^{-\frac{d+1}{2}} \in \mathcal{L}_1$ and

$$\|x(1 - \Delta)^{-\frac{d+1}{2}}\|_1 \leq c_d \|x\|_{W^{d,1}}.$$

(b) $x(1 - \Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$ and

$$\|x(1 - \Delta)^{-\frac{d}{2}}\|_{1,\infty} \leq c_d \|x\|_{W^{d,1}}.$$

3. INTEGRATION FORMULA MODULO A CONSTANT FACTOR

For every $\phi \in L_\infty(\mathbb{R}^d)$, we define a bounded operator $T_\phi : L_2(\mathbb{R}_\theta^d) \rightarrow L_2(\mathbb{R}_\theta^d)$ by the formula

$$T_\phi : \int_{\mathbb{R}^d} f(s)U(s)ds \rightarrow \int_{\mathbb{R}^d} f(s)\phi(s)U(s)ds, \quad f \in L_2(\mathbb{R}^d).$$

Lemma 3.1. If ϕ is a Schwartz function, then $T_\phi : L_1(\mathbb{R}_\theta^d) \rightarrow L_1(\mathbb{R}_\theta^d)$.

Proof. We claim that

$$T_\phi x = \int_{\mathbb{R}^d} (\mathcal{F}\phi)(u)U(-\theta^{-1}u)xU(\theta^{-1}u)du, \quad x \in L_2(\mathbb{R}_\theta^d).$$

Since both sides above define bounded operators on $L_2(\mathbb{R}_\theta^d)$ and since the set $\{\text{Op}(f) : f \text{ is Schwartz}\}$ is dense in $L_2(\mathbb{R}_\theta^d)$, it suffices to establish the claim for

$$x = \int_{\mathbb{R}^d} f(s)U(s)ds, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Using the inverse Fourier transform, we write

$$\phi(s) = \int_{\mathbb{R}^d} (\mathcal{F}\phi)(u)e^{i\langle u,s \rangle}du, \quad s \in \mathbb{R}^d.$$

Since both f and $\mathcal{F}\phi$ are Schwartz functions, it follows that

$$T_\phi x = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f(s)(\mathcal{F}\phi)(u)e^{i\langle u,s \rangle}U(s)dsdu.$$

It follows from (2) that

$$e^{i\langle u,s \rangle}U(s) = U(-\theta^{-1}u)U(s)U(\theta^{-1}u).$$

Therefore,

$$T_\phi x = \int_{\mathbb{R}^d} (\mathcal{F}\phi)(u) \left(\int_{\mathbb{R}^d} f(s) U(-\theta^{-1}u) U(s) U(\theta^{-1}u) ds \right) du.$$

Using the definition of x , we obtain

$$\int_{\mathbb{R}^d} f(s) U(-\theta^{-1}u) U(s) U(\theta^{-1}u) ds = U(-\theta^{-1}u) x U(\theta^{-1}u).$$

This proves the claim.

Now, we prove the assertion of the lemma as follows.

$$\|T_\phi x\|_1 \leq \int_{\mathbb{R}^d} |(\mathcal{F}\phi)(u)| \cdot \|U(-\theta^{-1}u) x U(\theta^{-1}u)\|_1 du = \|\mathcal{F}\phi\|_1 \|x\|_1.$$

□

Lemma 3.2. *For every $x \in W^{d,1}(\mathbb{R}_\theta^d)$, the mapping*

$$t \rightarrow U(-t)xU(t), \quad t \in \mathbb{R}^d,$$

is a continuous $W^{d,1}(\mathbb{R}_\theta^d)$ -valued function. Moreover,

$$\|U(-t)xU(t)\|_{W^{d,1}} = \|x\|_{W^{d,1}}.$$

Proof. It follows from Leibniz rule that

$$\begin{aligned} [D_k, U(-t)xU(t)] &= [D_k, U(-t)] \cdot xU(t) + U(-t) \cdot [D_k, x] \cdot U(t) + U(-t)x \cdot [D_k, U(t)] = \\ &= -t_k U(-t)xU(t) + U(-t)[D_k, x]U(t) + t_k U(-t)xU(t) = U(-t)[D_k, x]U(t). \end{aligned}$$

Iterating the latter inequality, we obtain

$$\partial^\alpha(U(-t)xU(t)) = U(-t)\partial^\alpha(x)U(t).$$

Thus,

$$\begin{aligned} \|U(-t)xU(t)\|_{W^{d,1}} &= \sum_{|\alpha| \leq d} \|\partial^\alpha(U(-t)xU(t))\|_1 = \\ &= \sum_{|\alpha| \leq d} \|U(-t)\partial^\alpha(x)U(t)\|_1 = \sum_{|\alpha| \leq d} \|\partial^\alpha(x)\|_1 = \|x\|_{W^{d,1}}. \end{aligned}$$

We now establish the continuity. For every $y \in \mathcal{L}_1$, the mapping

$$t \rightarrow V(-t)yV(t), \quad t \in \mathbb{R}^d,$$

is continuous in the \mathcal{L}_1 -norm whenever the mapping $t \rightarrow V(t)$ is strongly continuous. Recall that $(L_\infty(\mathbb{R}_\theta^d), \tau_\theta)$ is $*$ -isomorphic (so that trace is preserved) to $(\mathcal{L}(L_2(\mathbb{R}^{\frac{d}{2}})), \text{Tr})$. Thus, the mapping

$$t \rightarrow U(-t)\partial^\alpha(x)U(t) = \partial^\alpha(U(-t)xU(t))$$

is continuous in L_1 -norm. This completes the proof. □

Lemma 3.3. (a) *If f is Schwartz, then $\text{Op}(f) \in W^{d,1}(\mathbb{R}_\theta^d)$.*

(b) *The set $\{\text{Op}(f) : f \text{ is Schwartz}\}$ is dense in $L_1(\mathbb{R}_\theta^d)$. In particular, $W^{d,1}(\mathbb{R}_\theta^d)$ is dense in $L_1(\mathbb{R}_\theta^d)$.*

Proof. There exists a sequence $\{e_{kl}\}_{k,l \geq 0} \subset L_\infty(\mathbb{R}_\theta^d)$ such that

- (i) $e_{k_1 l_1} e_{k_2 l_2} = \delta_{l_1, k_2} e_{k_1 l_2}$ and $e_{kl}^* = e_{lk}$.
- (ii) $\tau_\theta(e_{kk}) = 1$.
- (iii) $\sum_{k \geq 0} e_{kk} = 1$ in strong operator topology.
- (iv) for every $k, l \geq 0$, there exists a Schwartz function f_{kl} such that $e_{kl} = \text{Op}(f_{kl})$.

The existence of such a sequence is established in Lemma 2.4 in [8] (see also additional references therein). A particular formula for f_{kl} can be found on p. 618 in [8] in terms of Laguerre polynomials.

We prove (a). Let f be a Schwartz function. By Proposition 2.5 in [8], one can write f as

$$f = \sum_{k,l \geq 0} c_{kl} f_{kl}, \quad \sum_{k,l \geq 0} |c_{kl}| < \infty.$$

Thus,

$$\text{Op}(f) = \sum_{k,l \geq 0} c_{kl} e_{kl},$$

where the series converges in L_1 -norm. Thus, $\text{Op}(f) \in L_1(\mathbb{R}_\theta^d)$. Let $f_\alpha(t) = t^\alpha f(t)$, $t \in \mathbb{R}^d$. By (4), $\partial^\alpha(\text{Op}(f)) = \text{Op}(f_\alpha)$. Since f_α is also a Schwartz function, it follows that $\partial^\alpha(\text{Op}(f)) \in L_1(\mathbb{R}_\theta^d)$. This proves (a).

To prove (b), note that, for every $x \in L_1(\mathbb{R}_\theta^d)$,

$$\sum_{k,l \leq N} e_{kk} x e_{ll} = \left(\sum_{k \leq N} e_{kk} \right) x \left(\sum_{l \leq N} e_{ll} \right) \rightarrow x$$

in \mathcal{L}_1 -norm as $N \rightarrow \infty$. Note that $e_{kk} x e_{ll}$ is a scalar multiple of $e_{kl} = \text{Op}(f_{kl})$. Since a linear combination of Schwartz functions is again a Schwartz function, it follows that

$$\sum_{k,l \leq N} e_{kk} x e_{ll} \in \{\text{Op}(f) : f \text{ is Schwartz}\} \subset W^{d,1}(\mathbb{R}_\theta^d).$$

This proves (b). □

Lemma 3.4. *If F is a continuous functional on $W^{d,1}(\mathbb{R}_\theta^d)$ such that*

$$F(x) = F(U(-t)xU(t)), \quad x \in W^{d,1}(\mathbb{R}_\theta^d), \quad t \in \mathbb{R}^d,$$

then $F = \tau_\theta$ (up to a constant factor).

Proof. Let $T : W^{d,1}(\mathbb{R}_\theta^d) \rightarrow W^{d,1}(\mathbb{R}_\theta^d)$ be defined by setting

$$Tx = \int_{\mathbb{R}^d} U(-\theta^{-1}t)xU(\theta^{-1}t)e^{-\frac{1}{2}|t|^2} dt.$$

The integral is understood as a Bochner integral of a continuous $W^{d,1}(\mathbb{R}_\theta^d)$ -valued function (the continuity and convergence of the integral follow from Lemma 3.2).

For every $x \in W^{d,1}(\mathbb{R}_\theta^d)$, we have

$$F(Tx) = \int_{\mathbb{R}^d} F(U(-\theta^{-1}t)xU(\theta^{-1}t))e^{-\frac{1}{2}|t|^2} dt = \int_{\mathbb{R}^d} F(x)e^{-\frac{1}{2}|t|^2} dt = (2\pi)^{\frac{d}{2}} F(x).$$

Thus,

$$F(x) = (2\pi)^{-\frac{d}{2}} F(Tx), \quad x \in W^{d,1}(\mathbb{R}_\theta^d).$$

We claim that $\|Tx\|_{W^{d,1}} \leq c_d \|x\|_1$ for every $x \in W^{d,1}(\mathbb{R}_\theta^d)$. To see this, let

$$x = \int_{\mathbb{R}^d} f(s)U(s)ds, \quad f \in L_2(\mathbb{R}^d).$$

If, in the proof of Lemma 3.1, we select $\phi(t) = e^{-\frac{1}{2}|t|^2}$, $t \in \mathbb{R}^d$, then the argument given there yields

$$Tx = \int_{\mathbb{R}^d} f(s)U(s)e^{-\frac{1}{2}|s|^2} ds.$$

By (4), we have

$$\partial^\alpha(Tx) = \int_{\mathbb{R}^d} f(s)U(s)s^\alpha e^{-\frac{1}{2}|s|^2} ds.$$

Let $\phi_\alpha(s) = s^\alpha e^{-\frac{1}{2}|s|^2}$, $s \in \mathbb{R}^d$. We have that $\partial^\alpha \circ T = T_{\phi_\alpha}$. By Lemma 3.1, $T_{\phi_\alpha} : L_1(\mathbb{R}_\theta^d) \rightarrow L_1(\mathbb{R}_\theta^d)$ is a bounded operator. This proves the claim.

For every $x \in W^{d,1}(\mathbb{R}_\theta^d)$, we have

$$|F(x)| = (2\pi)^{-\frac{d}{2}} |F(Tx)| \leq (2\pi)^{-\frac{d}{2}} \|F\|_{(W^{d,1})^*} \|Tx\|_{W^{d,1}} \leq c_d \|F\|_{(W^{d,1})^*} \|x\|_1.$$

Thus, a functional F on $W^{d,1}(\mathbb{R}_\theta^d)$ is bounded in $\|\cdot\|_1$ -norm. By the Hahn-Banach Theorem, F extends to a bounded functional on $L_1(\mathbb{R}_\theta^d)$. Hence, there exists $y \in L_\infty(\mathbb{R}_\theta^d)$ such that

$$F(x) = \tau_\theta(xy), \quad x \in W^{d,1}(\mathbb{R}_\theta^d).$$

Clearly,

$$F(U(-t)xU(t)) = \tau_\theta(U(-t)xU(t)y) = \tau_\theta(xU(t)yU(-t)).$$

Comparing the last 2 equalities, we obtain

$$\tau_\theta(xU(t)yU(-t)) = \tau_\theta(xy), \quad x \in W^{d,1}(\mathbb{R}_\theta^d).$$

Since $W^{d,1}(\mathbb{R}_\theta^d)$ is dense in $L_1(\mathbb{R}_\theta^d)$, it follows that $y = U(t)yU(-t)$ for every $t \in \mathbb{R}^d$. In other words, y commutes with every $U(t)$ and, therefore, with every element in $L_\infty(\mathbb{R}_\theta^d)$. Since $L_\infty(\mathbb{R}_\theta^d)$ is a factor (see Theorem 2.6), it follows that y is a scalar operator. This completes the proof. \square

The following proposition is a light version of Theorem 1.1.

Proposition 3.5. *If $x \in W^{d,1}(\mathbb{R}_\theta^d)$, then $x(1 - \Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$ and*

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = c_\varphi \tau_\theta(x)$$

for every continuous trace on $\mathcal{L}_{1,\infty}$ and for some constant c_φ .

Proof. By Theorem 2.10 (b), the functional

$$F : x \rightarrow \varphi(x(1 - \Delta)^{-\frac{d}{2}}), \quad x \in W^{d,1}(\mathbb{R}_\theta^d),$$

is a well defined bounded linear functional on $W^{d,1}(\mathbb{R}_\theta^d)$.

Since φ is unitarily invariant, it follows that

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle t, \nabla \rangle} x(1 - \Delta)^{-\frac{d}{2}} e^{-i\langle t, \nabla \rangle}), \quad t \in \mathbb{R}^d.$$

By the Spectral Theorem, we have

$$(1 - \Delta)^{-\frac{d}{2}} e^{-i\langle t, \nabla \rangle} = e^{-i\langle t, \nabla \rangle} (1 - \Delta)^{-\frac{d}{2}},$$

and so

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle t, \nabla \rangle} x e^{-i\langle t, \nabla \rangle} (1 - \Delta)^{-\frac{d}{2}}).$$

For every $s \in \mathbb{R}^d$, we have (see (5))

$$e^{i\langle t, \nabla \rangle} U(s) e^{-i\langle t, \nabla \rangle} = e^{i\langle t, s \rangle} U(s).$$

On the other hand, it follows from (2) that

$$U(-\theta^{-1}t)U(s)U(\theta^{-1}t) = e^{i\langle t, s \rangle} U(s).$$

Comparing preceding equalities, we arrive at

$$e^{i\langle t, \nabla \rangle} U(s) e^{-i\langle t, \nabla \rangle} = U(-\theta^{-1}t)U(s)U(\theta^{-1}t).$$

It follows that

$$e^{i\langle t, \nabla \rangle} x e^{-i\langle t, \nabla \rangle} = U(-\theta^{-1}t) x U(\theta^{-1}t), \quad x \in L_\infty(\mathbb{R}_\theta^d).$$

Combining the preceding paragraphs, we obtain

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(U(-\theta^{-1}t) x U(\theta^{-1}t) (1 - \Delta)^{-\frac{d}{2}}).$$

Applying Lemma 3.4 to our functional F , we conclude the argument. \square

4. PROOF OF MEASURABILITY

Lemma 4.1. *If $K \in W^{2d+2,1}([0, 1]^d \times [0, 1]^d)$ and if $T : L_2((0, 1)^d) \rightarrow L_2((0, 1)^d)$ is an integral operator with integral kernel K , then $T \in \mathcal{L}_1$ and $\|T\|_1 \leq c_d \|K\|_{W^{2d+2,1}}$.*

Proof. Let $K \in W^{2d+2,1}([-\pi, \pi]^d \times [-\pi, \pi]^d)$ be an extension of K such that

$$\|K\|_{W^{2d+2,1}([-\pi, \pi]^d \times [-\pi, \pi]^d)} \leq c_d \|K\|_{W^{2d+2,1}([0, 1]^d \times [0, 1]^d)}$$

and such that K vanishes on and near the boundary. Thus, $K \in W^{2d+2,1}(\mathbb{T}^d \times \mathbb{T}^d)$. Let $S : L_2(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ be an integral operator with integral kernel K . We have $T = M_{\chi_{(0,1)^d}} S M_{\chi_{(0,1)^d}}$. Thus, $\|T\|_1 \leq \|S\|_1$.

Let us write Fourier series

$$K(t, s) = \sum_{m_1, m_2 \in \mathbb{Z}^d} c_{m_1, m_2} e_{m_1}(t) e_{m_2}(s), \quad t, s \in \mathbb{T}^d.$$

Set

$$S_{m_1, m_2} \xi = \langle \xi, e_{-m_2} \rangle e_{m_1}, \quad \xi \in L_2(\mathbb{T}^d).$$

It is an integral operator on $L_2(\mathbb{T}^d)$ with the integral kernel $(t, s) \rightarrow e_{m_1}(t) e_{m_2}(s)$. Hence,

$$S = \sum_{m_1, m_2 \in \mathbb{Z}^d} c_{m_1, m_2} S_{m_1, m_2}.$$

By triangle inequality, we have

$$\begin{aligned} \|S\|_1 &\leq \sum_{m_1, m_2 \in \mathbb{Z}^d} |c_{m_1, m_2}| \leq \\ &\leq \sup_{m_1, m_2 \in \mathbb{Z}^d} (1 + |m_1|^2 + |m_2|^2)^{d+1} |c_{m_1, m_2}| \cdot \sum_{m_1, m_2 \in \mathbb{Z}^d} (1 + |m_1|^2 + |m_2|^2)^{-d-1}. \end{aligned}$$

Observe that $(1 + |m_1|^2 + |m_2|^2)^{d+1} c_{m_1, m_2}$ is the (m_1, m_2) -th Fourier coefficient of the function $(1 - \Delta_{\mathbb{T}^{2d}})^{d+1}(K)$ (here, $\Delta_{\mathbb{T}^{2d}}$ is the Laplacian on the torus \mathbb{T}^{2d}). Taking into account that Fourier coefficients do not exceed the L_1 -norm, we infer that

$$(1 + |m_1|^2 + |m_2|^2)^{d+1} |c_{m_1, m_2}| \leq (2\pi)^{-2d} \|(1 - \Delta_{\mathbb{T}^{2d}})^{d+1} K\|_1 \leq c_d \|K\|_{W^{2d+2,1}}.$$

Here, the last inequality follows from the definition of a Sobolev space. \square

In what follows, we consider the tensor product of 2 bounded operators on a Hilbert space H as a bounded operator on the Hilbert space $H \bar{\otimes} H$.

Lemma 4.2. *If $T \in \mathcal{L}_{1,\infty}$ and $S \in \mathcal{L}_1$, then $S \otimes T \in \mathcal{L}_{1,\infty}$ and*

$$(6) \quad \varphi(S \otimes T) = \text{Tr}(S) \cdot \varphi(T)$$

for every continuous trace φ on $\mathcal{L}_{1,\infty}$.

Proof. Firstly, we show that $S \otimes T \in \mathcal{L}_{1,\infty}$. Let $z(t) = t^{-1}$, $t > 0$. By definition, we have $\mu(T) \leq \|T\|_{1,\infty} z$. The crucial fact that $\mu(S \otimes z) = \|S\|_1 z$ is proved on p. 211 in [13]. Thus,

$$\|S \otimes T\|_{1,\infty} = \|S \otimes \mu(T)\|_{1,\infty} \leq \|T\|_{1,\infty} \|S \otimes z\|_{1,\infty} = \|T\|_{1,\infty} \|S\|_1.$$

We now turn to the proof of (6). If S is a rank one projection, then there is nothing to prove. If S is a positive finite rank operator, then the assertion follows by linearity. If S is an arbitrary finite rank operator, then the assertion again follows by linearity.

Let $S \in \mathcal{L}_1$ be arbitrary. Fix $\epsilon > 0$ and choose $S_1, S_2 \in \mathcal{L}_1$ such that $S = S_1 + S_2$, S_1 is finite rank and $\|S_2\|_1 \leq \epsilon$. Clearly,

$$\begin{aligned} \varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T) &= \\ &= (\varphi(S_1 \otimes T) - \text{Tr}(S_1) \cdot \varphi(T)) + (\varphi(S_2 \otimes T) - \text{Tr}(S_2) \cdot \varphi(T)). \end{aligned}$$

By the preceding paragraph, the summand in the first bracket vanishes. Thus,

$$\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T) = \varphi(S_2 \otimes T) - \text{Tr}(S_2) \cdot \varphi(T).$$

Hence,

$$\begin{aligned} |\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T)| &\leq |\varphi(S_2 \otimes T)| + |\text{Tr}(S_2) \cdot \varphi(T)| \leq \\ &\leq \|\varphi\|_{\mathcal{L}_{1,\infty}^*} \cdot (\|S_2 \otimes T\|_{1,\infty} + |\text{Tr}(S_2)| \|T\|_{1,\infty}). \end{aligned}$$

By the norm estimate in the first paragraph and by the assumption on S_2 , we have

$$|\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T)| \leq 2\epsilon \|\varphi\|_{\mathcal{L}_{1,\infty}^*} \|T\|_{1,\infty}.$$

Since $\epsilon > 0$ is arbitrarily small, the assertion follows. \square

In the following lemma, we consider the direct sum of bounded operators on a Hilbert space H as a bounded operator on a Hilbert space $\bigoplus_{m \geq 0} H$.

Lemma 4.3. *If the operators $\{T_m\}_{m \geq 0}$ are pairwise orthogonal, i.e. $T_{m_1} T_{m_2} = T_{m_1}^* T_{m_2} = 0$ for $m_1 \neq m_2$, then $\sum_{m \geq 0} T_m$ is unitarily equivalent⁴ to $\bigoplus_{m \geq 0} T_m$. Here, the sums are taken in the weak operator topology.*

Proof. Let p_1 and p_2 be projections on H . Since $t \rightarrow t^{\frac{1}{n}}$, $t > 0$, is an operator monotone function for every $n \geq 1$, it follows that

$$p_1 = p_1^{\frac{1}{n}} \leq (p_1 + p_2)^{\frac{1}{n}} \xrightarrow{\text{supp}} \text{supp}(p_1 + p_2).$$

Similarly, $p_2 \leq \text{supp}(p_1 + p_2)$ and, therefore,

$$p_1 \vee p_2 \leq \text{supp}(p_1 + p_2).$$

This simple fact can be also found in Proposition 2.5.14 in [10].

Let $p_m = \text{supp}(T_m)$ and $q_m = \text{supp}(T_m^*)$. It follows from the assumption that $p_{m_1} p_{m_2} = p_{m_1} q_{m_2} = q_{m_1} q_{m_2} = 0$, $m_1 \neq m_2$. Set $r_m = p_m \vee q_m$. We have

$$(p_{m_1} + q_{m_1})(p_{m_2} + q_{m_2}) = 0, \quad m_1 \neq m_2.$$

Thus,

$$\text{supp}(p_{m_1} + q_{m_1}) \cdot \text{supp}(p_{m_2} + q_{m_2}) = 0, \quad m_1 \neq m_2.$$

⁴To be pedantic, $\sum_{m \geq 0} T_m$ is unitarily equivalent to the direct sum $\bigoplus_{m \geq 0} T_m|_{r_m(H) \rightarrow r_m(H)}$, where r_m is the projection defined in the proof of Lemma 4.3. Clearly, T_m is unitarily equivalent to the direct sum $T_m|_{r_m(H) \rightarrow r_m(H)} \oplus 0_{(1-r_m)(H) \rightarrow (1-r_m)(H)}$. Thus, a direct sum $\bigoplus_{m \geq 0} T_m$ is unitarily equivalent to $(\sum_{m \geq 0} T_m) \oplus 0$. In what follows, we ignore this subtle difference and write unitary equivalence as stated in Lemma 4.3.

By the preceding paragraph, we have $r_{m_1}r_{m_2} = 0$, $m_1 \neq m_2$.

If $T = \sum_{m \geq 0} T_m$, then $r_m T = T_m$ and $Tr_m = T_m$ for every $m \geq 0$. Thus, $T = \bigoplus_{m \geq 0} T_m$, where T_m acts on the Hilbert space $r_m(H)$. \square

Let

$$h(t) = (1 + \sum_{k=1}^d [t_k]^2)^{-\frac{d}{2}}, \quad t \in \mathbb{R}^d.$$

The following proposition yields a special case of Theorem 1.1.

Proposition 4.4. *If f is a Schwartz function supported on $[-1, 1]^d$ and if $x = \text{Op}(f)$, then $xh(\nabla)$ is measurable.*

Proof. Step 1: We have that $xh(\nabla)$ is an integral operator with the kernel

$$K : (t, s) \rightarrow f(t - s)h(s)e^{\frac{i}{2}\langle s, \theta t \rangle}, \quad t, s \in \mathbb{R}^d.$$

By assumption on f , we have that

$$f(s - t) = 0, \quad s \in m_1 + [0, 1]^d, \quad t \in m_2 + [0, 1]^d, \quad m_1 - m_2 \notin \{-1, 0, 1\}^d.$$

Thus,

$$xh(\nabla) = \sum_{l_1, l_2 \in \{-1, 0, 1\}^d} T_{l_1, l_2}, \quad T_{l_1, l_2} = \sum_{\substack{m \in \mathbb{Z}^d \\ m \equiv l_2 \pmod{3}}} h(m) T_{m, l_1},$$

where T_{m, l_1} is an integral operator whose integral kernel is given by the formula

$$(t, s) \rightarrow f(t - s)e^{\frac{i}{2}\langle s, \theta t \rangle} \chi_{m+l_1+[0, 1]^d}(t) \chi_{m+[0, 1]^d}(s), \quad t, s \in \mathbb{R}^d,$$

Step 2: We claim that $T_{l_1, l_2} \in \mathcal{L}_{1, \infty}$ and is measurable.

Note that the operators $\{T_{m, l_1}\}_{\substack{m \in \mathbb{Z}^d \\ m \equiv l_2 \pmod{3}}}$ are pairwise orthogonal. Therefore, we have (\sim denotes unitary equivalence)

$$T_{l_1, l_2} \sim \bigoplus_{\substack{m \in \mathbb{Z}^d \\ m \equiv l_2 \pmod{3}}} (1 + |m|^2)^{-\frac{d}{2}} T_{m, l_1}.$$

By definition, $T_{m, l_1} : L_2(m + [-1, 2]^d) \rightarrow L_2(m + [-1, 2]^d)$. Define a unitary operator

$$U_m : L_2([-1, 2]^d) \rightarrow L_2(m + [-1, 2]^d)$$

by setting

$$(U_m \xi)(t) = e^{\frac{i}{2}\langle m, \theta t \rangle} \xi(t - m), \quad \xi \in L_2([-1, 2]^d), \quad t \in m + [-1, 2]^d.$$

Define an operator $S_{l_1} : L_2([-1, 2]^d) \rightarrow L_2([-1, 2]^d)$ to be an integral operator with the integral kernel

$$(t, s) \rightarrow f(t - s)e^{\frac{i}{2}\langle s, \theta t \rangle} \chi_{l_1+[0, 1]^d}(t) \chi_{[0, 1]^d}(s), \quad t, s \in [-1, 2]^d.$$

A direct computational argument shows that⁵

$$T_{m, l_1} = U_m S_{l_1} U_m^{-1}.$$

⁵Indeed,

$$(U_m^{-1} \xi)(t) = e^{-\frac{i}{2}\langle m, \theta t \rangle} \xi(t + m), \quad \xi \in L_2(m + [-1, 2]^d), \quad t \in [-1, 2]^d.$$

Thus,

$$(S_{l_1} U_m^{-1} \xi)(t) = \chi_{l_1+[0, 1]^d}(t) \cdot \int_{[0, 1]^d} f(t - s) e^{\frac{i}{2}\langle s, \theta(t+m) \rangle} \xi(s + m) ds.$$

Hence,

$$T_{l_1, l_2} \sim \bigoplus_{\substack{m \in \mathbb{Z}^d \\ m = l_2 \bmod 3}} (1 + |m|^2)^{-\frac{d}{2}} S_{l_1} \sim S_{l_1} \otimes \left\{ (1 + |m|^2)^{-\frac{d}{2}} \right\}_{m = l_2 \bmod 3}.$$

By Lemma 4.1, $S_{l_1} \in \mathcal{L}_1$. The claim follows now from Lemma 4.2. \square

Proof of Theorem 1.1. Choose a Schwartz function f_0 supported on $[-1, 1]^d$ such that $f_0(0) \neq 0$ and let $x_0 = \text{Op}(f_0)$. Set

$$k(t) = (1 + |t|^2)^{\frac{d+1}{2}} \cdot \left((1 + |t|^2)^{-\frac{d}{2}} - (1 + \sum_{k=1}^d [t_k]^2)^{-\frac{d}{2}} \right), \quad t \in \mathbb{R}^d.$$

Clearly, k is a bounded function on \mathbb{R}^d .

By Lemma 3.3 (a), we have $x_0 \in W^{d,1}(\mathbb{R}_\theta^d)$. Using the obvious equality

$$x_0(1 - \Delta)^{-\frac{d}{2}} - x_0 h(\nabla) = x_0(1 - \Delta)^{-\frac{d+1}{2}} \cdot k(\nabla)$$

and Theorem 2.10 (a), we infer that

$$x_0(1 - \Delta)^{-\frac{d}{2}} - x_0 h(\nabla) \in \mathcal{L}_1.$$

By Proposition 4.4, we have that $x_0 h(\nabla)$ is measurable and, hence, so is the operator $x_0(1 - \Delta)^{-\frac{d}{2}}$.

Let now $x \in W^{d,1}(\mathbb{R}_\theta^d)$ be arbitrary. Since f_0 is a Schwartz function, it follows that

$$\tau_\theta(x_0) = f_0(0) \neq 0.$$

Without loss of generality, $\tau_\theta(x_0) = 1$. Let $z = x - \tau_\theta(x)x_0 \in W^{d,1}(\mathbb{R}_\theta^d)$. Clearly, $\tau_\theta(z) = 0$. We have

$$\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(z(1 - \Delta)^{-\frac{d}{2}}) + \tau_\theta(x) \cdot \varphi(x_0(1 - \Delta)^{-\frac{d}{2}}).$$

By Proposition 3.5, the first summand vanishes. By the preceding paragraph, the second summand does not depend on φ . This completes the proof. \square

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Thus,

$$\begin{aligned} (U_m S_{l_1} U_m^{-1} \xi)(t) &= \chi_{l_1 + [0,1]^d}(t - m) \cdot \int_{[0,1]^d} e^{\frac{i}{2} \langle m, \theta t \rangle} f(t - s - m) e^{\frac{i}{2} \langle s, \theta t \rangle} \xi(s + m) ds = \\ &= \chi_{m + l_1 + [0,1]^d}(t) \cdot \int_{m + [0,1]^d} f(t - s) e^{\frac{i}{2} \langle s, \theta t \rangle} \xi(s) ds. \end{aligned}$$

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